Entropy Production in QCD

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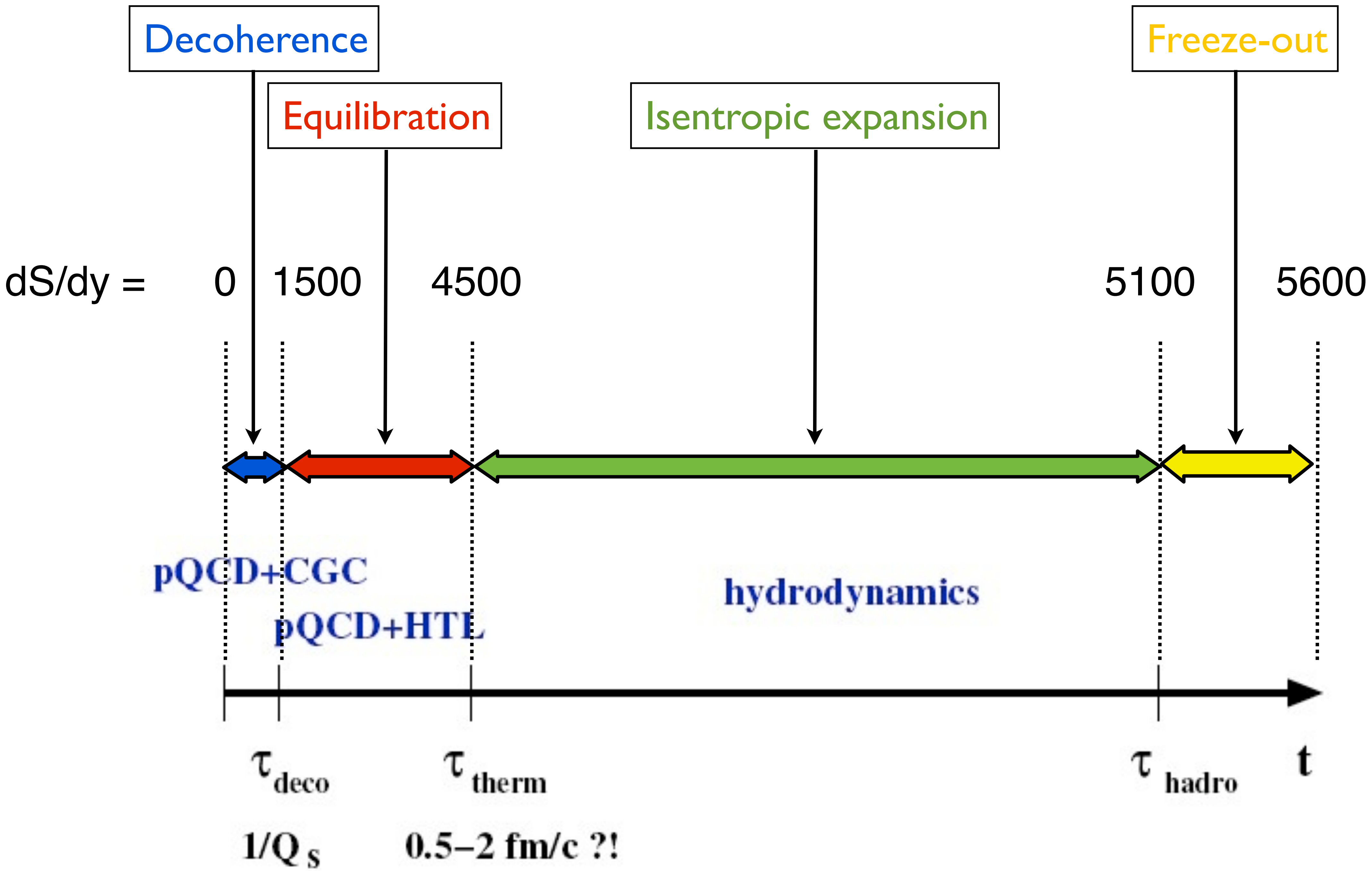
Based on:

T. Kunihiro, B.M., A. Ohnishi & A. Schäfer, PTP 121, 555 (2009) [0809.4831]

Entropic history



Final entropy

Bjorken's formula

$$s(\tau) : \frac{dN(\tau)/dy}{dV(\tau)/dy} \le \frac{(dN/dy)_{\text{final}}}{\pi R^2 \tau}$$

Phase space analysis (Pal & Pratt):

$$\left.\frac{dS}{dy}\right|_{\text{final}} = \sum_i \int \frac{d^3r\,d^3p}{(2\pi)^3 dy}\left[-f_i \ln f_i \pm (1 \pm f_i)\ln(1 \pm f_i)\right]$$

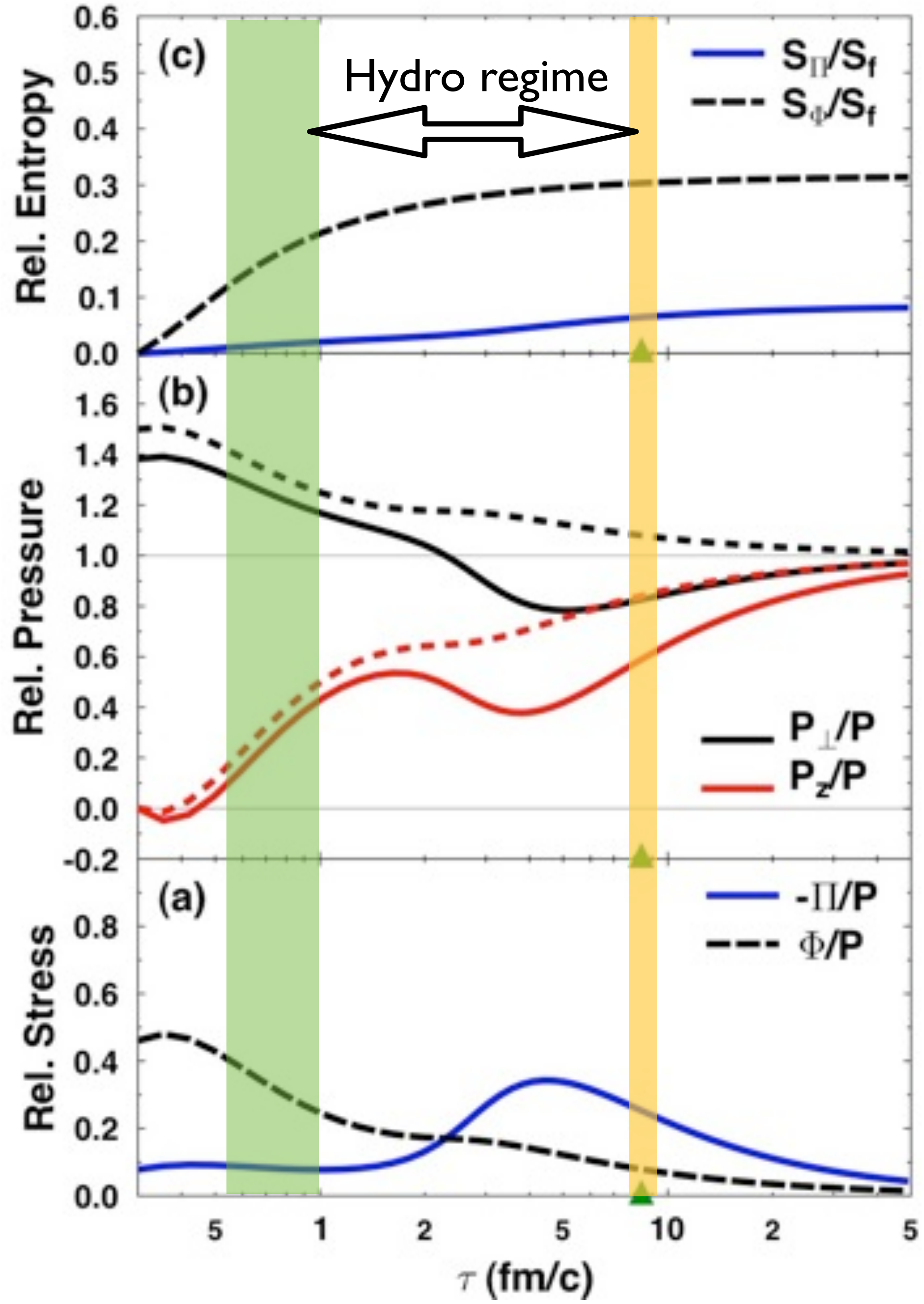
$$= 5600 \pm 500 \qquad \text{[for 6\% central Au+Au @ 200]}$$

Chemical analysis (BM & Rajagopal):

$$\left.\frac{dS}{dy}\right|_{\text{final}} = \sum_i (S/N)_i \frac{dN_i}{dy} = 5100 \pm 200 \qquad \text{[for same cond.]}$$

The 10% increase during hadronic expansion is compatible with moderate viscosity of the hot hadronic gas phase. Quantitative study with hadronic Boltzmann cascade would be desirable.

Viscous hydrodynamics



$$P_T = P_{eq} + \Pi + \frac{1}{2}\Phi$$

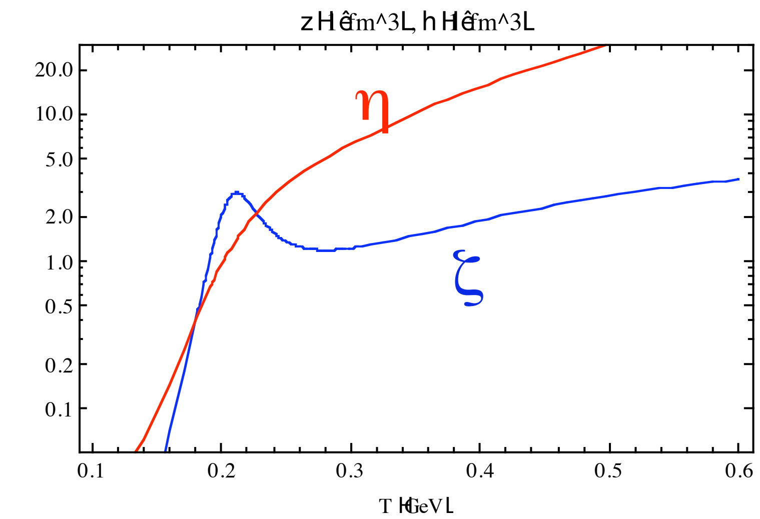
$$P_T = P_{eq} + \Pi - \Phi$$

R. Fries, BM, A. Schäfer, PRC 78, 034913 (2008)

$$\tau_s \frac{d\Phi}{d\tau} = \frac{4\eta}{3\tau} - \left(1 + \frac{4\tau_s}{3\tau}\right)\Phi - \frac{\lambda_1}{2\eta^2}\Phi^2$$

$$\tau_b \frac{d\Pi}{d\tau} = \frac{\varsigma}{\tau} - \Pi$$

$\eta / s,\ \varsigma / s$ from lattice



Lattice EOS

$$\tau_s = \tau_b = \frac{2 - \ln 2}{2\pi T} \quad (N = 4\ \mathrm{SUSY})$$

Conclusion: 10-20% increase of *S* likely in hydrodynamic flow regime.

Decoherence

Complete decoherence of coherent state generates: $S_{\text{deco}} \approx \frac{1}{2}\left(\ln 2\pi\bar{n} + 1\right)$

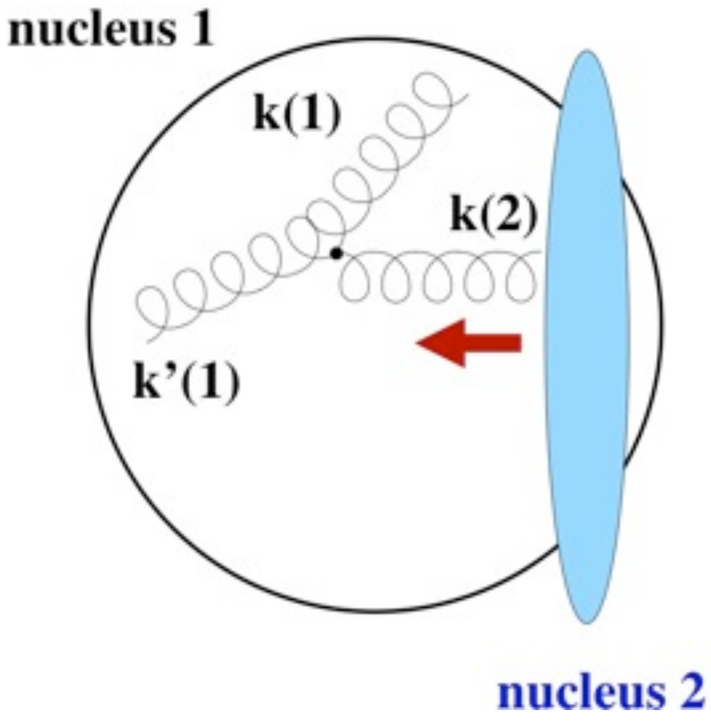
Application to CGC initial state, counting causally disconnected transverse domains:

$$\frac{dS_{\text{deco}}}{dy} \approx \frac{1}{2} Q_s^2 R^2 \alpha_s \left(\ln \frac{2\pi}{\alpha_s} + 1 \right) \sim 1{,}500 \quad (\text{for } \alpha_s \approx 3).$$

Decoherence time can be calculated from:

$$\frac{\mathrm{Tr}\,\hat{\rho}^2(t)}{\left(\mathrm{Tr}\,\hat{\rho}(t)\right)^2} \sim \exp\left(-t/\tau_{\text{deco}}\right)$$

$\longrightarrow$ $\boxed{\tau_{\text{deco}} = c\,Q_s^{-1} \quad \text{with} \quad c \approx 1}$



BM & A. Schäfer, PRC 73 (2006) 054905; R. Fries, BM, A. Schäfer, arXiv:0807.1093 (PRC in print).

The need

Conclusion: About 50% of final entropy may be attributed to (transverse) decoherence (~30%), hydrodynamic expansion (~10%), and hadronic freezeout (~10%).

The remainder must be due to pre-equilibrium dynamics of the “glasma”.

Needed: A systematic approach to computing the transition from decohering initial color fields to quasi-equilibrium describable by hydrodynamics.

The problem

The von Neumann entropy $S_{\mathrm{vN}} = -\mathrm{Tr}\left[\rho \ln \rho\right]$

is conserved for any closed quantum system described by a Hamiltonian.

Approach 1: For system *X* interacting with its environment *Y*, the reduced entropy

$$S_X = -\mathrm{Tr}_X\left[\rho_X \ln \rho_X\right] \quad \text{with} \quad \rho_X = \mathrm{Tr}_Y\left[\rho\right]$$

increases as a result of growing entanglement between *X* and *Y*.
Possible approach: Consider, a rapidity interval Δy as "system" and the remainder as "environment", which cannot effectively communicate due to causality.

Problem: Entanglement entropy usually proportional to surface area, not volume.

Approach 2: Consider the effective growth of the entropy due to the increasing intrinsic complexity of the quantum state after "coarse graining".

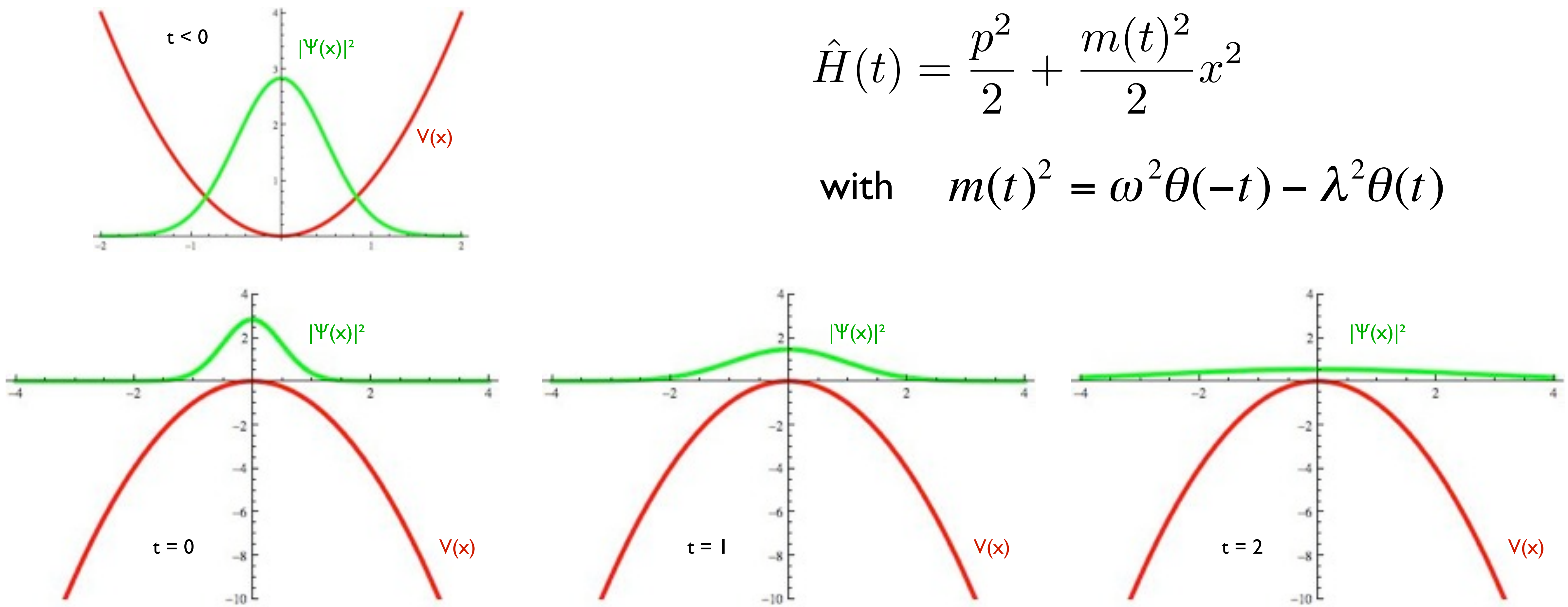
Problem: How to coarse grain without assuming the answer ?

The “pencil on its tip”

The decay of an unstable vacuum state is a common problem, e.g., in cosmology and in condensed matter physics. Paradigm case: inverted oscillator.

$$\hat{H}(t) = \frac{p^2}{2} + \frac{m(t)^2}{2}x^2$$

with $m(t)^2 = \omega^2\theta(-t) - \lambda^2\theta(t)$

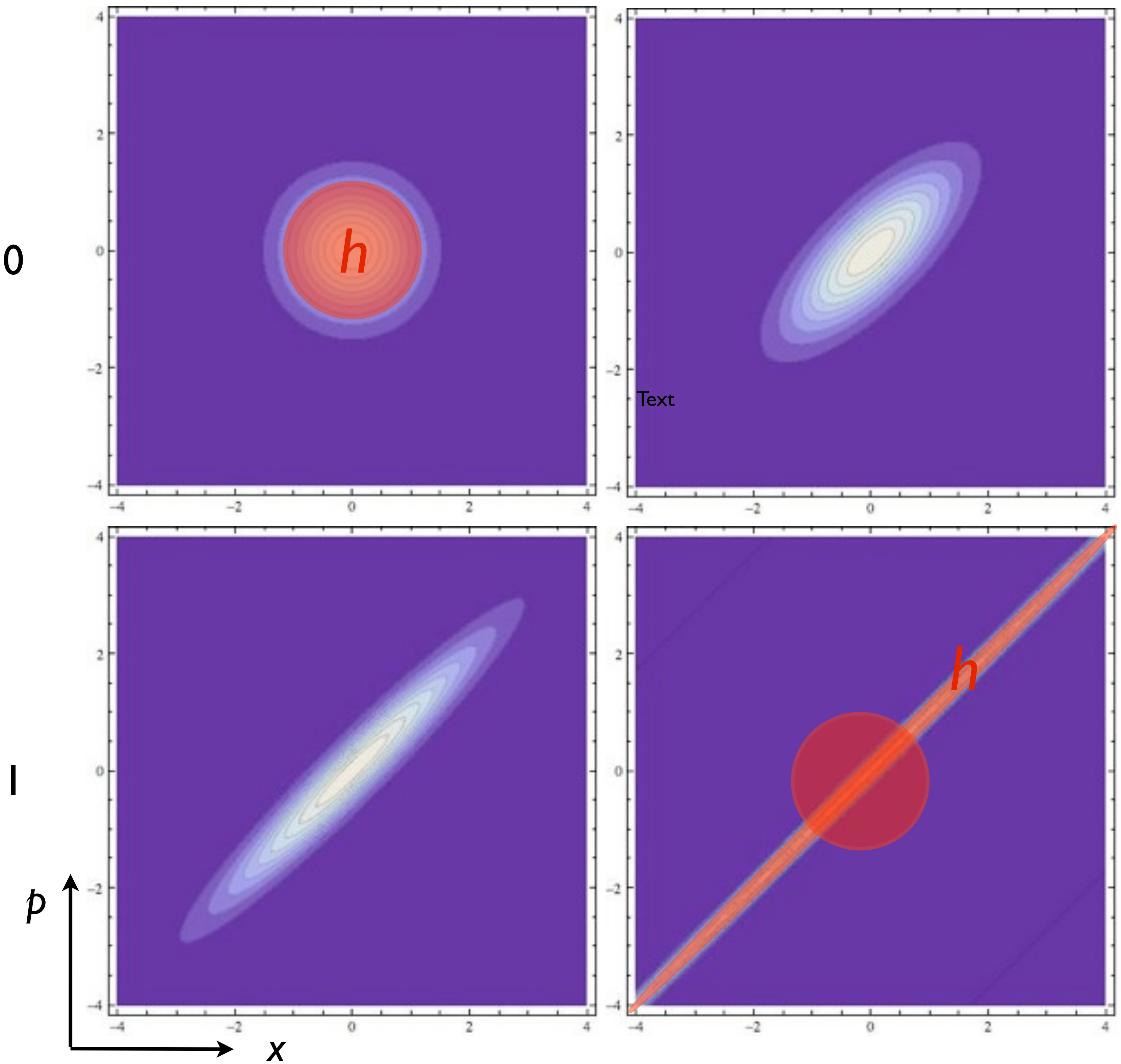


Wigner function: $$W(q,p;t) = \int du\; e^{-ipu} \langle q + \frac{1}{2}u|\; \hat{\rho}(t)\; |q - \frac{1}{2}u\rangle$$

Wigner function

$t = 0$

$t = 0.5$



$t = 1$

$t = 2$

Husimi transform

- Problem: Wigner function cannot be interpreted as a probability distribution, because *W*(*p,x*) is not positive (semi-)definite.
- Idea (*Husimi* - 1940): Smear the Wigner function with a Gaussian minimum-uncertainty wave packet:

$$H_\Delta(p,x;t) \equiv \int \frac{dp'\,dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(x-x')^2\right) W(p',x';t)$$

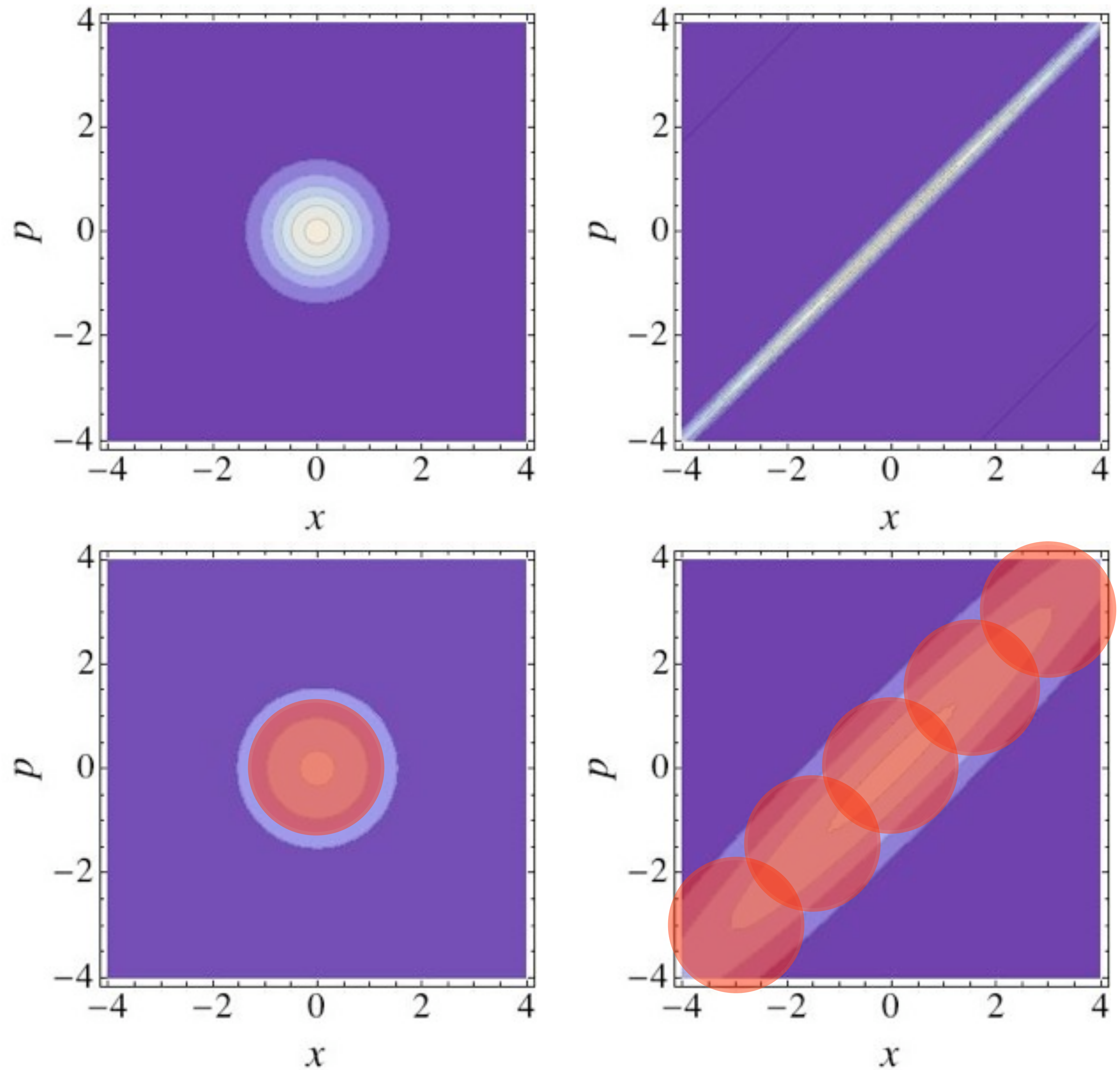
- *H*(*p*,*x*) can be shown to be the expectation value of the density matrix in a coherent oscillator state $|x+ip\rangle$ and thus *H*(*p*,*x*) ≥ 0 holds always.
- *H*(p,x) can be considered as a probability density, enabling the definition of a minimally coarse grained entropy (*Wehrl* - 1978):

$$S_{\mathrm{H},\Delta}(t) = -\int \frac{dp\,dx}{2\pi\hbar} H_\Delta(p,x;t) \ln H_\Delta(p,x;t)$$

Wigner vs. Husimi

Wigner function

$t = 0$



$t = 2$

Husimi function

$t = 0$

$t = 2$

S_H entropy growth

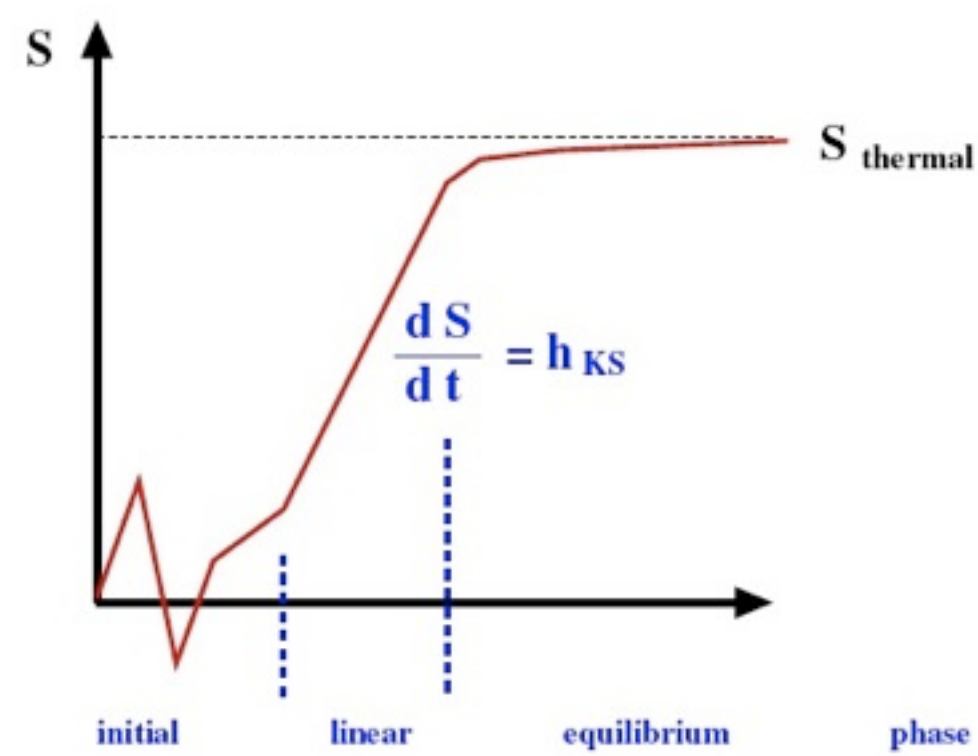
$$\frac{dS_H}{dt} = \frac{\lambda\sigma\rho\sinh 2\lambda t}{\sigma\rho\cosh 2\lambda t + 1 + \delta'\delta} \xrightarrow{t\to\infty} \lambda$$ with $\rho,\sigma,\delta,\delta'$ constants dep. on ω,λ

but independent of Δ and $\hbar$!!!

Many modes: $$\frac{dS_H}{dt} \xrightarrow{t\to\infty} \sum_k \lambda_k\,\theta(\lambda_k)$$

This is the Kolmogorov-Sinai (KS) entropy h_{KS} known from classical dynamical system theory.

KS-entropy describes the growth rate of the entropy for a coarse grained phase space density in the approach toward ergodic equilibrium.
[see e.g.: Latora & Baranger, PRL 82 (1999) 520.]



From Quantum Mechanics to Quantum Field Theory:

The Wigner Functional

Wigner functional

Adapt the phase space formulation to the field representation of quantum field theory in order make the classical field limit more transparent.

Wigner functional = Adaptation to QFT. Start with a scalar quantum field Φ.

Position space representation:

$$W[\Phi(x),\Pi(x);t] \quad = \quad \int \mathcal{D}\varphi(x)\,\exp\left[-i\int dx\;\Pi(x)\varphi(x)\right]\;\langle\Phi(x)+\frac{1}{2}\varphi(x)|\;\hat{\rho}(t)\;|\Phi(x)-\frac{1}{2}\varphi(x)\rangle$$

Momentum space representation:

$$W[\Phi(p),\Pi(p);t] = \int \mathcal{D}\varphi(p)\,\exp\left[-i\int_0^\infty dp\;\left(\Pi^*(p)\varphi(p)+\Pi(p)\varphi^*(p)\right)\right]\langle\Phi(p)+\frac{1}{2}\varphi(p)|\;\hat{\rho}(t)\;|\Phi(p)-\frac{1}{2}\varphi(p)\rangle$$

[S. Mrówczyński & BM, Phys. Rev. D 50 (1994) 7452]

Unstable vacuum in QFT

$$\hat{H}(t) = \int_0^\infty \frac{dp}{2\pi}\left(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + (m^2(t)+p^2)\,\hat{\Phi}^\dagger(p)\hat{\Phi}(p)\right) \quad \text{with} \quad m^2(t) = m^2\,\theta(-t) - \mu^2\,\theta(t)$$

Split problem into stable ($p^2 > \mu^2$) and unstable ($p^2 < \mu^2$) modes.

Initial Wigner functional:

$$W[\Pi,\Phi;t] = C\,e^{-\int \frac{dp}{2\pi}\left(\frac{|\Pi_p|^2}{E_p} + E_p|\Phi_p|^2\right)} \quad \text{with} \quad E_p = \sqrt{p^2+m^2}$$

W is constant along a classical trajectory:

$$W[\Pi,\Phi;t] = C\,e^{-\int \frac{dp}{2\pi}\left(\frac{|\Pi_p^0|^2}{E_p} + E_p|\Phi_p^0|^2\right)}$$

$|p| < \mu$

$$\begin{aligned}\Phi_p^0 &= \Phi_p(t)\cosh\lambda_p t - \frac{\Pi_p(t)}{\lambda_p}\sinh\lambda_p t \\ \Pi_p^0 &= \Pi_p(t)\cosh\lambda_p t - \lambda_p\,\Phi_p(t)\sinh\lambda_p t\end{aligned}$$

$$\lambda_p = \sqrt{\mu^2 - p^2}$$

$|p| > \mu$

$$\begin{aligned}\Phi_p^0 &= \Phi_p(t)\cos\omega_p t - \frac{\Pi_p(t)}{\omega_p}\sin\omega_p t \\ \Pi_p^0 &= \Pi_p(t)\cos\omega_p t + \omega_p\,\Phi_p(t)\sin\omega_p t\end{aligned}$$

$$\omega_p = \sqrt{p^2 - \mu^2}$$

Entropy growth

Husimi functional:

$$H_\Delta[\Pi,\Phi;t] = \prod_{|p|<\mu} \frac{2}{\sqrt{A_p(t)}} \exp\left[-\frac{R(\Pi_p,\Phi_p;t)}{A_p(t)}\right] \times \prod_{|p|>\mu} \frac{2}{\sqrt{\tilde{A}_p(t)}} \exp\left[-\frac{\tilde{R}(\Pi_p,\Phi_p;t)}{\tilde{A}_p(t)}\right]$$

Husimi-Wehrl entropy:

$$S_{\mathrm{H},\Delta}(t) \;=\; \int \frac{D\Pi\, D\Phi}{2\pi} H_\Delta \ln H_\Delta = V \int_{|p|<\mu} \frac{dp}{2\pi}\left[\frac{1}{2}\ln\frac{A_p(t)}{4}+1\right] + V\int_{|p|>\mu}\frac{dp}{2\pi}\left[\frac{1}{2}\ln\frac{\tilde{A}_p(t)}{4}+1\right]$$

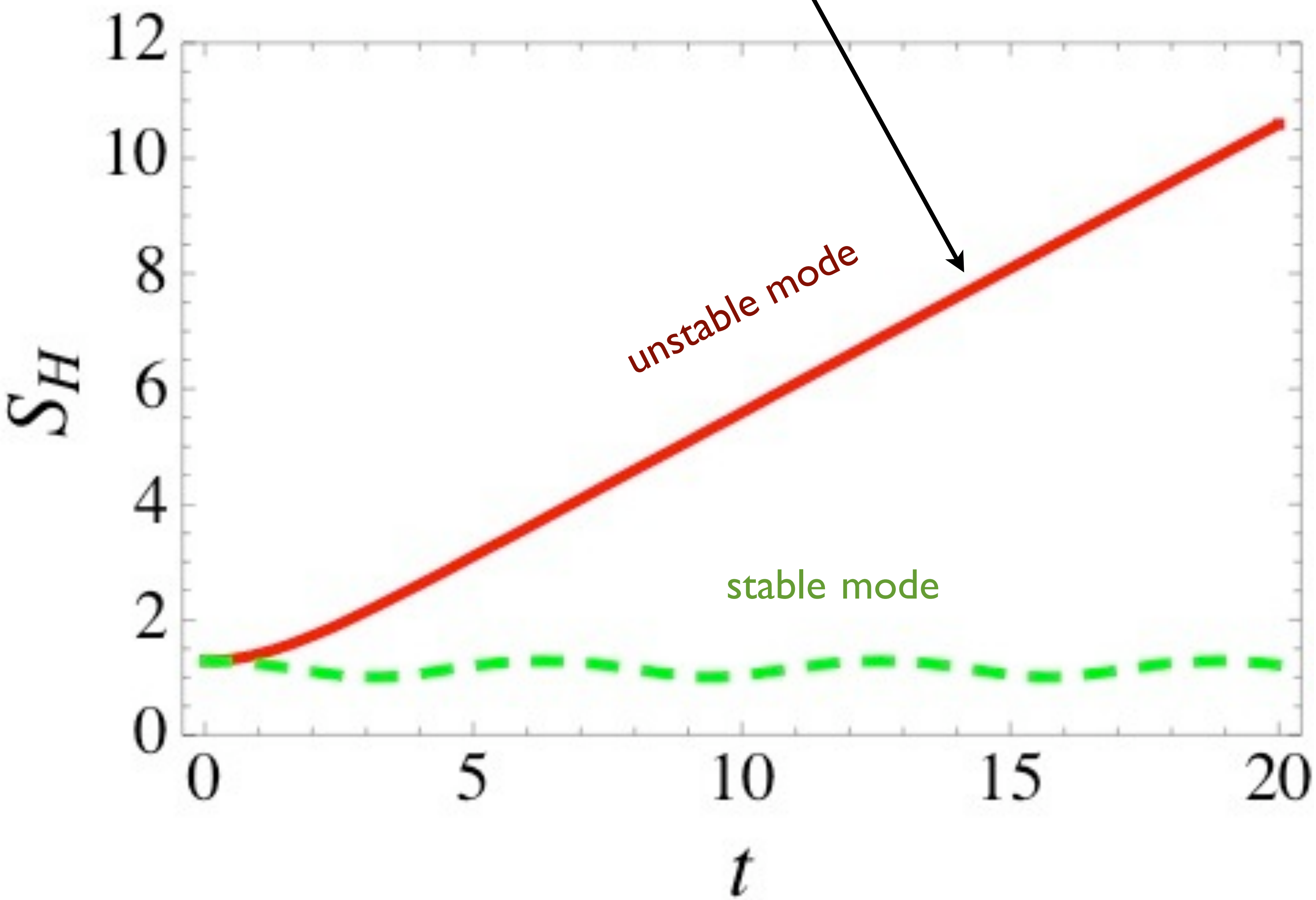
Growth rate:

$$\begin{aligned}\frac{dS_{\mathrm{H},\Delta}}{dt} \;&=\; V\int_{|p|<\mu}\frac{dp}{2\pi}\frac{\sigma_p(\Delta^2+\lambda_p^2)\sinh 2\lambda_p t}{A_p(t)\Delta} + V\int_{|p|>\mu}\frac{dp}{2\pi}\frac{\tilde{\delta}_p(\omega_p^2-\Delta^2)\sin 2\omega_p t}{\tilde{A}_p(t)\Delta} \\ &\xrightarrow{t\to\infty}\; V\int_{-\mu}^{\mu}\frac{dp}{2\pi}\lambda_p = \frac{V\mu^2}{8}.\end{aligned}$$

[Extensive (!) and independent of ħ]

Instability begets entropy

Only S_H of unstable modes grows !



Parametric resonance instability

Big Bang entropy

Other application: Reheating after cosmic inflation.

At the end of inflationary period, the scalar inflaton field falls out of its false vacuum state and begins to oscillate around the true minimum of its potential. Other fields coupling to the inflaton field now experience a periodically oscillating potential (or mass in quantum field theory). Model case: scalar field with bi-quadratic coupling.

$$\mathcal{L}(\hat{\chi}) = \frac{1}{2}\left(g^{\mu\nu}\frac{\partial\hat{\chi}}{\partial x^\mu}\frac{\partial\hat{\chi}}{\partial x^\nu} - g^2\Phi(t)^2\hat{\chi}^2\right)$$

$\Phi(t) = \Phi_0 \cos(\omega t)$ inflaton field

Canonical transformation: $(X_k, P_k) = (r_k \cos\alpha_k, \omega r_k \sin\alpha_k)$

Consider single mode *k*.

$$W(\alpha, n, \tau) \approx 2e^{-\alpha^2/\tilde{\pi}^2 - n^2\tilde{\pi}^2} \quad \text{with} \quad \tilde{\pi}(\tau) = \pi\, e^{-2\mu\tau}$$

$$H_\Delta(\alpha, n, \tau) \approx \frac{2\sqrt{\tilde{\pi}^2\Delta}}{1+\tilde{\pi}^2\Delta}\exp\left(-\frac{\alpha^2\Delta + n^2\tilde{\pi}^2}{1+\tilde{\pi}^2\Delta}\right)$$

$$\boxed{S_{\mathrm{H},\Delta}(\tau) \stackrel{\tau\to\infty}{\longrightarrow} 2\mu\tau + \mathrm{const.}}$$

Wigner vs. Husimi

Wigner function

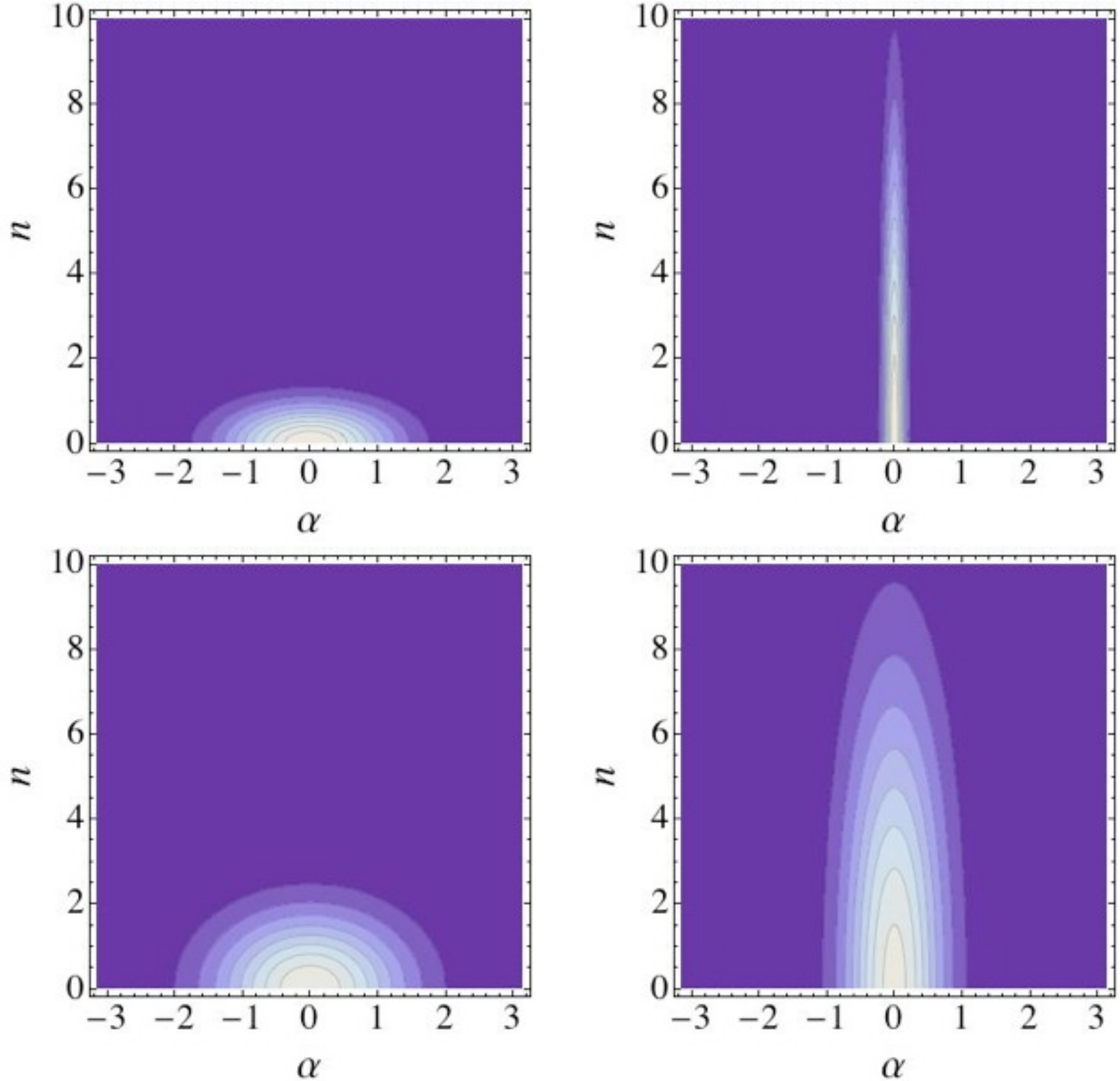
$t = 1$

$t = 3$

Husimi function

$t = 1$

$t = 3$



Yang-Mills Theory

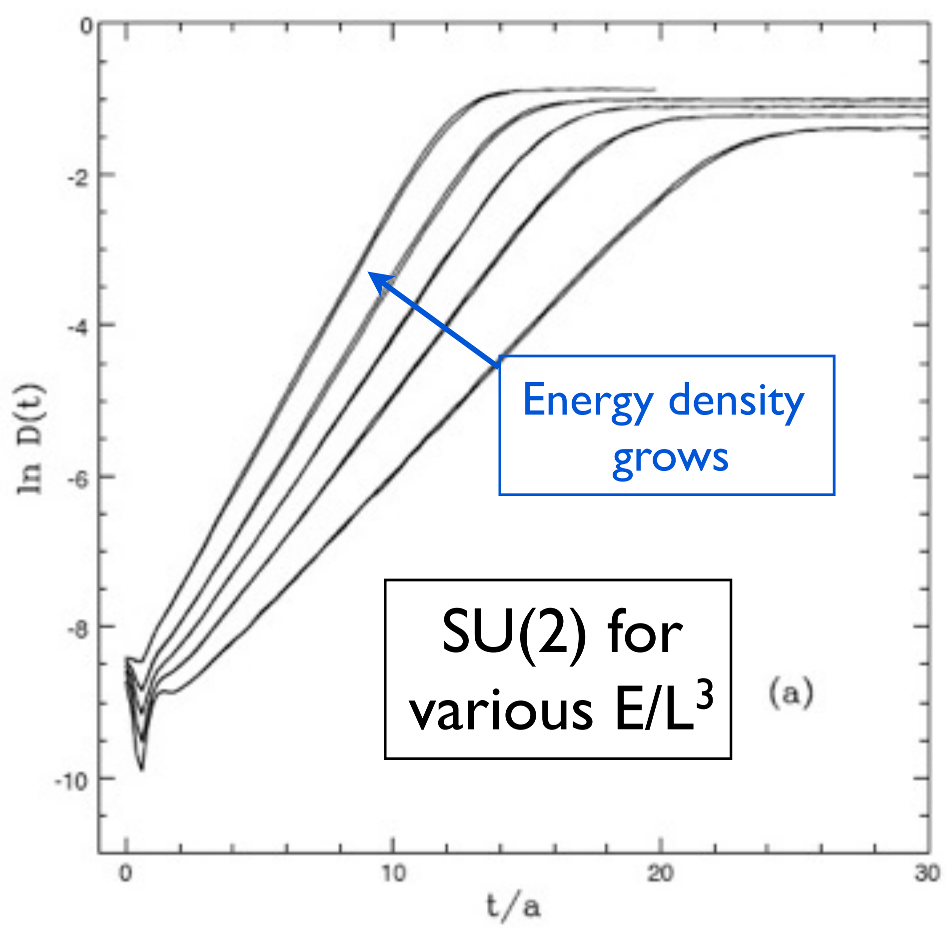
Yang-Mills Fields

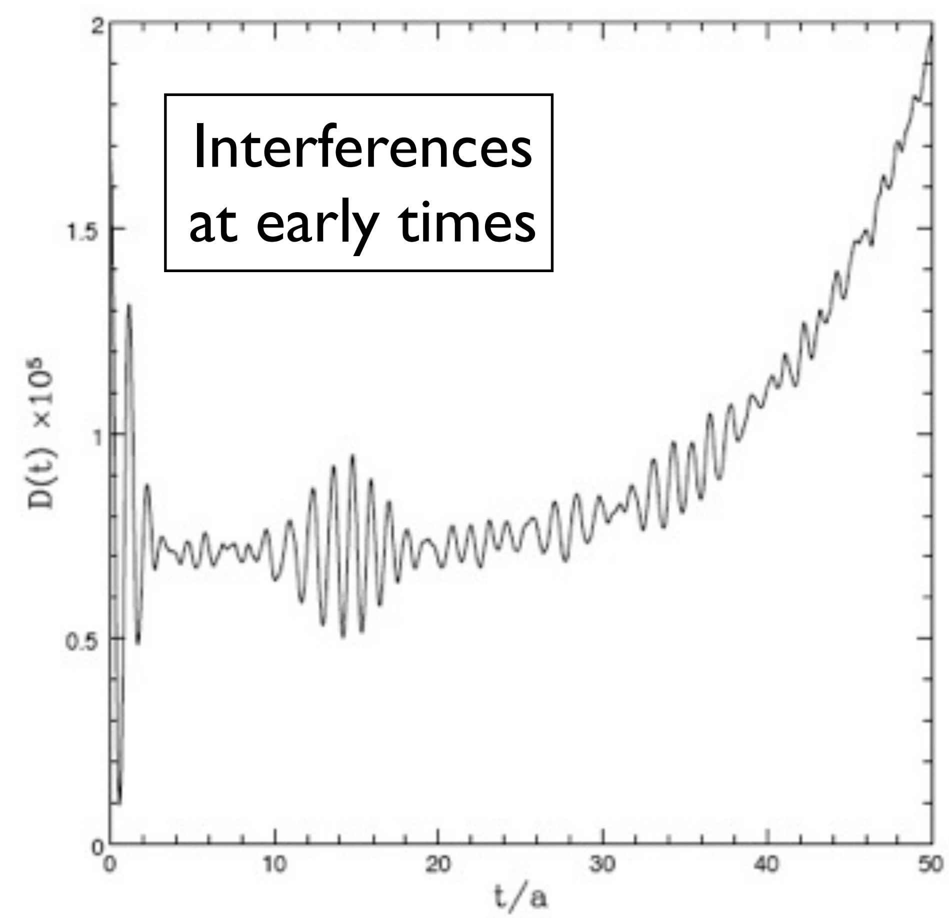
(Gauge invariant) distance measure needed:

$$\text{e.g.:} \quad D\left[A^{(1)},A^{(2)}\right] = \int d^3x \left|\varepsilon^{(1)}(x)^2 - \varepsilon^{(2)}(x)^2\right|$$

Yang-Mills Instabilities first observed in IR limit by S.G. Matinyan & G.K Savvidy (1981)

B. Müller & A. Trayanov, PRL 68 (92) 3387



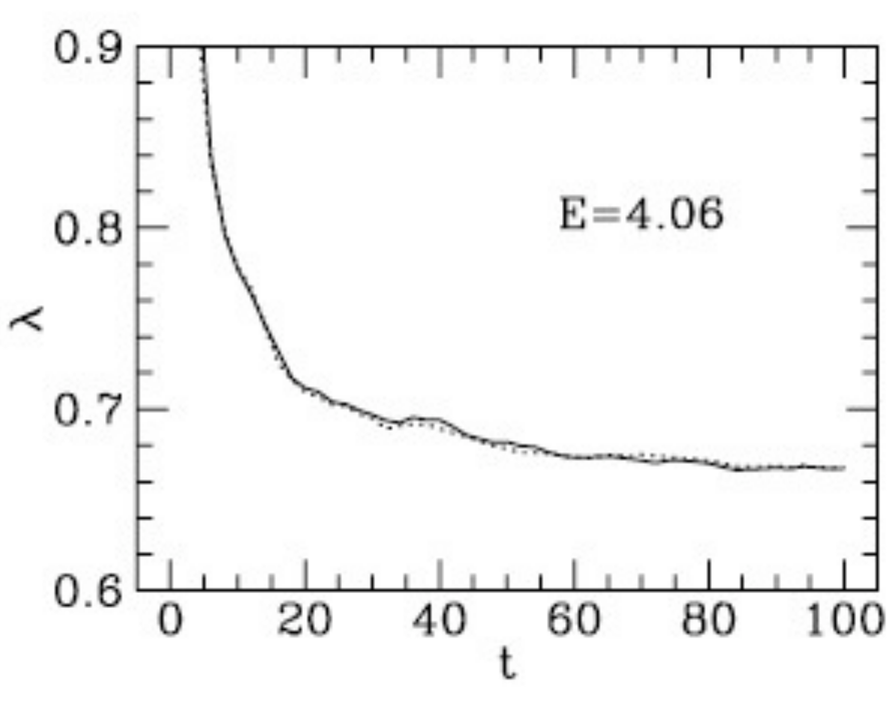


Lyapunov spectrum

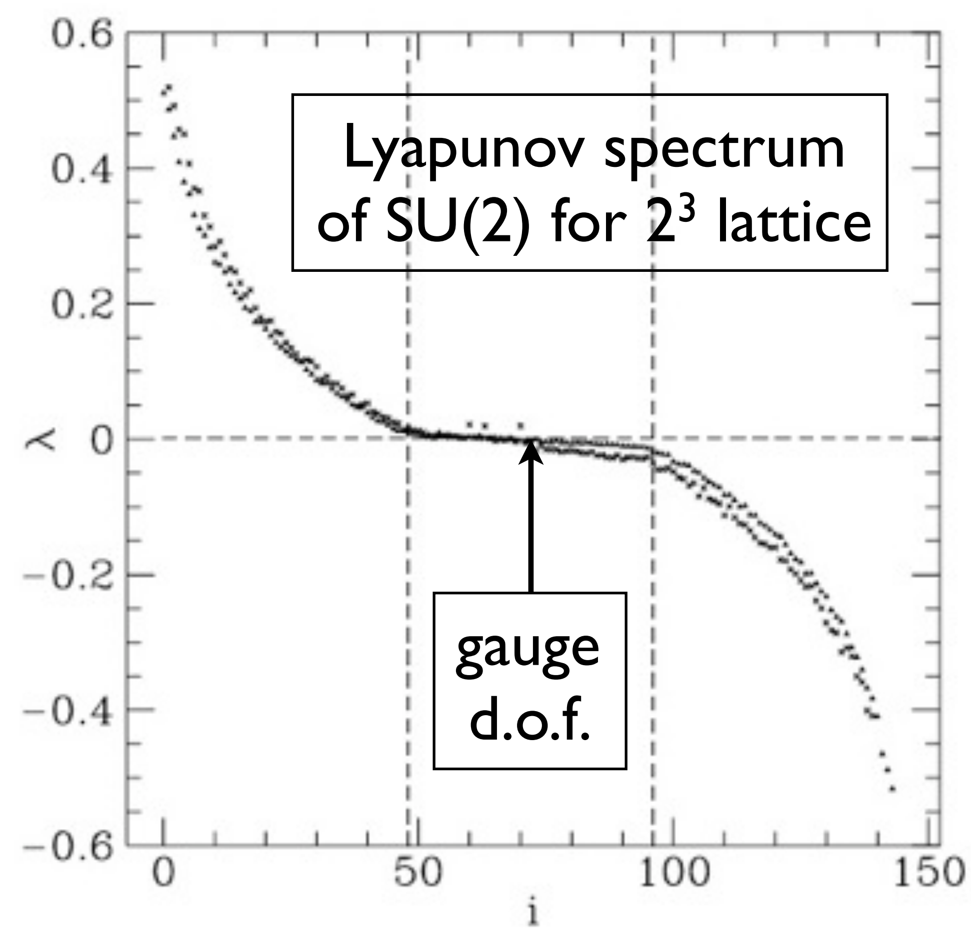
Systematic studies in T.S. Biro, C. Gong, B. Müller & A. Trayanov, IJMP C5 (94) 113

Rescaling method permits determination of complete spectrum of eigenvalues and eigenvectors the unstable modes.

Local instability exponents are larger than asymptotic Lyapunov exponents



C. Gong, PRD 49 (94) 2642

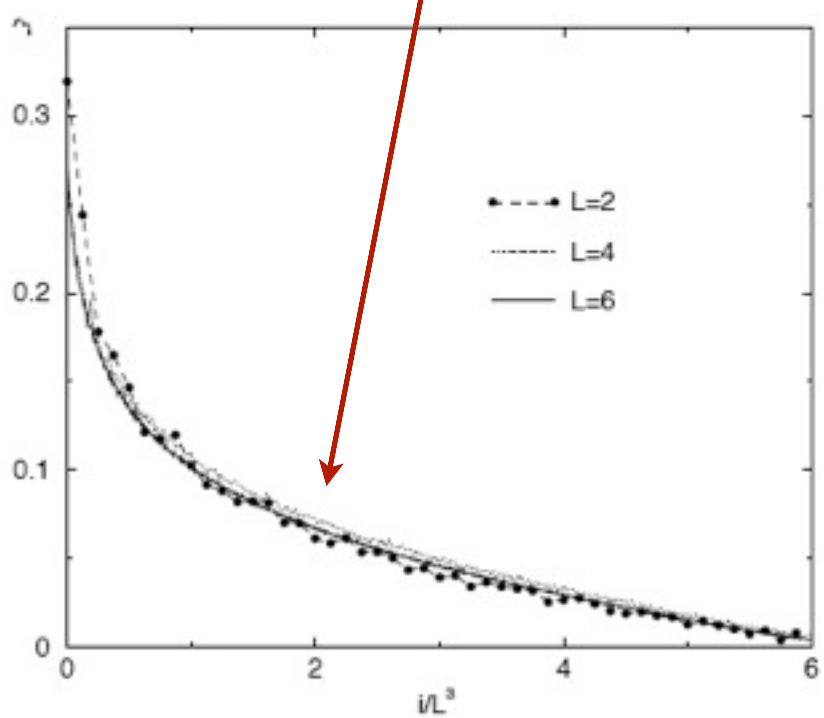


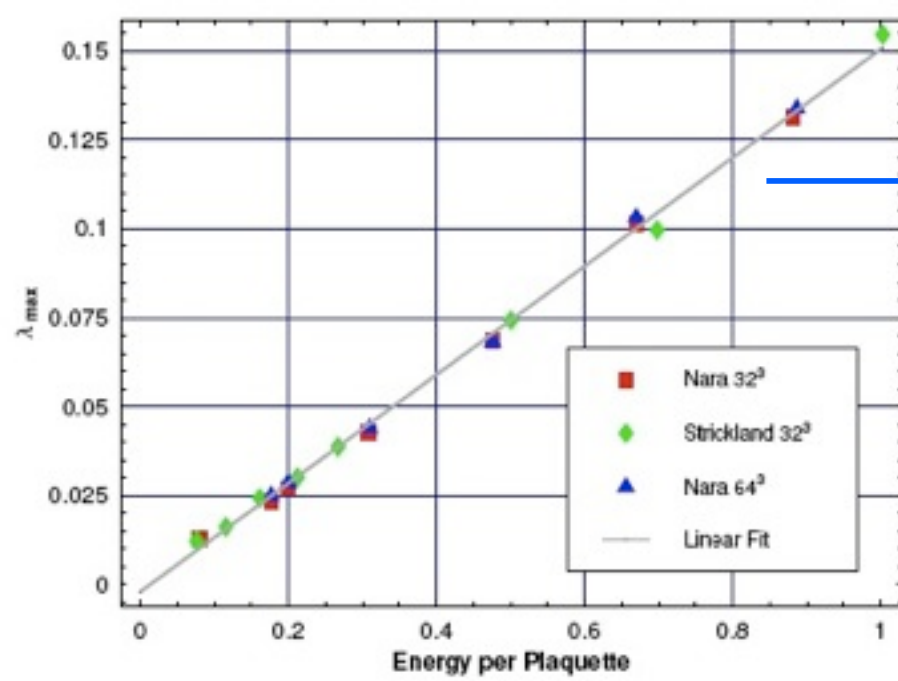
Yang-Mills fields

$$h_{\mathrm{KS}} = \sum_{\lambda_k > 0} \lambda_k \approx 0.079\,(18L^3)\,\lambda_{\max}$$

for SU(2) Hamiltonian LGT.

[J. Bolte, B.M., A. Schäfer, Phys. Rev D 61 (2000) 054506]





$$\lambda_{\max} \approx \frac{0.15\,g^2 E}{3L^3}$$

$$\Downarrow$$

$$h_{KS} \approx 0.07\,g^2 E$$

(extensive !)

$$\frac{dS}{dy\,d\tau} \approx 0.07\,\frac{4\pi\alpha_s}{h}\,\frac{4}{\pi}\,\frac{dE_T}{dy} \approx 1.1\alpha_s\,\frac{600\,\mathrm{GeV}}{h} \approx 1{,}000\,/\,(\mathrm{fm/c})$$

in central Au+Au (@ 200 GeV).

More systematic study of field configurations relevant to RHIC needed !

Beyond classical YM

Idea: Use generalized Gaussian wave packets in the link representation of lattice gauge fields [Gong, BM, Biró, Nucl. Phys. A568 (1994) 727].

Lattice (KS) Hamiltonian: $$H = \frac{g^2}{a}\left(\sum_l \frac{1}{2} E_l^a E_l^a + \frac{4}{g^4}\mathrm{Re}\sum_p (1 - \mathrm{tr}\, U_p)\right)$$

can be scaled to dimensionless variables:

$$ag^2 H \to H, \qquad t/a \to t, \qquad g^2 E \to E, \qquad g^2 \hbar \to \hbar$$

Wave packet *ansatz*:

$$\Phi[U_l] = \prod_l \phi_l(U_l) = \prod_l \frac{1}{\sqrt{N_l}} \exp\left(\frac{b_l}{2}\mathrm{tr}(U_l U_{l0}^{-1}) - \frac{1}{\hbar}\mathrm{tr}(E_{l0} U_l U_{l0}^{-1})\right)$$

Semiclassical evolution equation obtained from variational principle:

$$\delta \int_{t1}^{t2} \langle \Phi | (i\hbar\,\partial_t - H) | \Phi \rangle = 0$$ with respect to parameters: b_l (t), U_{l0} (t), E_{l0} (t).

Conclusions

Husimi functional provides a practical method to calculate the rate of (coarse grained) entropy growth in quantum field theory. Various applications:

- Decay of unstable vacuum states
- Decay of coherently oscillating excited states (e.g. reheating after inflation)
- Equilibration of QCD matter

Wigner functional method permits smooth interpolation between field eigenstates and particle excitations. This allows for the approximate treatment of quantum coherence and uncertainty relation effects.

Classical Yang-Mills theory on the lattice exhibits many local instabilities, implying an extensive KS entropy, which grows linearly with total energy.

Next goal: Study of the entropy growth rate of field configurations relevant to RHIC in *semiclassical* lattice gauge theory. Initial state Wigner functional in the CGC model has been constructed by Fukushima, Gelis & McLerran [Nucl. Phys. A786 (2007) 107].

TK, BM, AO, AS, Toru Takahashi & Arata Yamamoto, in progress - stay tuned.

The End !

Eqs. of motion

$$H_{\mathrm{eff}} = \sum_l \left[\frac{1}{2}(1-\frac{f(v_l)}{2v_l})E_{l0}^2 + \hbar^2\frac{3f(v_l)}{16v_l}|b_l|^2\right] + 4\sum_p (1-f_p\,U_{p0})$$

where $\quad b_l = v_l + iw_l \qquad f(v) = I_2(2v)/I_1(2v)$ and $f_p = \prod_{l\in p} f(v_l)$

$$\frac{dE_l^a}{dt} = \frac{i}{f(v_l)}\sum_{p(l)}(f_p\tau^a U_p) - \frac{3\hbar}{8}\frac{w_l}{v_l}E_l^a,$$

$$\frac{dU_l}{dt} = \frac{i}{2}\left(\frac{1}{f(v_l)} - \frac{1}{2v_l}\right)E_l U_l,$$

$$\frac{dv_l}{dt} = \frac{3\hbar}{8}\frac{f(v_l)}{f'(v_l)}\frac{w_l}{v_l},$$

$$\frac{dw_l}{dt} = \left(\frac{1}{f(v_l)} - \frac{1}{2v_l}\right)E_l^2 + \frac{E_l^2}{4v_l} + \frac{2}{f(v_l)}\sum_{p(l)} f_p U_p - \frac{3}{16}\hbar^2\left(v_l + \frac{w_l^2}{v_l}\right),$$

$$-\frac{f(v_l)}{f'(v_l)}\left(\frac{E_l^2}{4v_l^2} + \frac{3}{16}\hbar^2\left(1-\frac{w_l^2}{v_l^2}\right)\right),$$

Scaling implies:

$$\hbar \leftrightarrow g^2\hbar = 4\pi\alpha_s.$$